# Introduction

In the quest to mimic the precision of an experienced right fielder's throw or to optimize fluid flow within pipes, we as emerging engineers are tasked with numerical challenges that not only demand accuracy but also an understanding of practical applications in our field. This project takes us on a journey through the application of computational methods to analyze trajectories and forces in a real-world context. First, we calculate the ideal initial angle for a baseball throw, given a set of initial conditions, ensuring the ball's path from the fielder's hand to the catcher is both precise and practical. Next, we delve into the electrostatic force exerted by a charged ring on a point charge, a problem that tests our understanding of electric field distributions. Finally, we tackle the complexities of fluid dynamics by computing the Fanning friction factor for different Reynolds numbers, using the von Karman equation. Through the use of various numerical methods—ranging from Bisection to Muller's method—we not only derive solutions but also evaluate the methods themselves for their efficiency and accuracy, ensuring our calculations are within a tight error threshold of 0.01%. This report details our approach, findings, and insights gained from employing these methods in solving each problem, highlighting the synergy between theoretical knowledge and its practical application in engineering.

# Project Structure

## Main Functions for Numerical Solutions of Equations

This project was coded in Octave with the following functionality:

`menu\_options` - The script functions as an interactive tool that repeatedly presents a menu to the user, allowing them to choose from various numerical methods for root-finding. Upon selection, the user is prompted to enter necessary parameters and the chosen algorithm proceeds to calculate an approximation of the root of a mathematical function. The core of the script is designed to facilitate the application of different algorithms like Bisection and Newton's method, each utilizing its own iterative procedure and convergence criteria. The process continues in a loop, providing results for each method until the user opts to exit, ensuring a user-driven experience without restarting the script.

` bisection\_method\_project1` - This script implements the bisection method, a numerical technique for finding roots of a continuous function. The method requires an initial bracket, given by xl and xu, within which the root is believed to lie, and iteratively narrows down this bracket to pinpoint the root's location. The user supplies the function for which the root is sought, an acceptable error tolerance es, and a maximum number of iterations `maxit`.

At the start, the function checks whether the function values at the initial bracket boundaries have opposite signs—a necessary condition for the presence of a root. If this condition isn't met, it halts with an error.

The iterative process begins, calculating the midpoint of the current bracket and evaluating the function at that point. If the function value is zero or the error tolerance is met (the relative difference between the current and previous midpoints is less than es), the function concludes the root has been found. Otherwise, it decides which half of the bracket contains the root by checking the sign of the function at the midpoint, then repeats the process with the narrowed bracket.

The algorithm continues until the error tolerance is satisfied or the maximum number of iterations is reached. It records the midpoint estimates and the approximate relative errors at each step, returning these lists as outputs—`xr` for the root estimates and `ea` for the errors. If the maximum number of iterations is reached without meeting the error tolerance, it notifies the user that the root was not found to the desired precision.

` false\_position\_method\_project1` - This scripted function applies the false position (or “regula falsi”) method to find a root of a given function. This method is an improvement over the bisection method in that it potentially converges faster. It uses a linear interpolation to approximate the root, taking into account the function values at the bracketing interval.

The user provides an initial guess for the lower (xl) and upper (xu) bounds of the interval, the function (`func`) for which the root is sought, an allowable tolerance (es), and a maximum number of iterations (`maxit`).

The function starts by evaluating the function at the initial guesses to ensure that they bracket a root (i.e., the function values have different signs). If not, an error is displayed and execution stops.

For each iteration up to `maxit`, the method calculates an approximation of the root (xi) by interpolating the line between the points (xl, f(xl)) and (xu, f(xu)) and finding where it crosses the x-axis. After each new approximation, it updates the error estimate (`ea`) and checks if the function value at the new approximation is close enough to zero or if the approximation is within the desired tolerance. If so, it stops and declares convergence.

If the function value at the new approximation has a different sign from the value at the lower bound, it replaces the upper bound with the approximation (and vice versa). This maintains the bracket around the root.

When the loop ends, either from finding a satisfactory root or reaching the maximum number of iterations, it returns the last approximation of the root (`xr`) and the corresponding error estimate (`ea`). If the loop finishes because the maximum number of iterations was reached without satisfying the error tolerance, it indicates that the root has not been located within the desired accuracy.

`fixed\_point\_iteration\_project1` - The scripted function implements the Fixed-Point Iteration method, used to approximate solutions of the equation f(x) = 0. This method assumes that the equation can be rewritten in the form x = g(x), so the root finding problem becomes an iteration process starting from an initial guess x\_0.

The user supplies the initial guess x\_0, the function f (passed as a string f\_func\_str) which needs to be solved, a corresponding function g (passed as a string g\_func\_str) used for iterations, a tolerance level e\_a, and a maximum number of iterations i.

The function initializes the iteration counter and an empty list to keep track of all approximate root values (x\_vals) and the corresponding error estimates (ea\_vals). For each iteration, it applies the iteration function g to the current approximation, checks the value of the function f at this new point to see if it is sufficiently close to zero (within the tolerance), and records the error. If the stopping criteria are met—either the function value is within the error tolerance or the maximum number of iterations is reached—the process halts.

At each step, it calculates the relative error based on the current and previous approximations and stores it. This process repeats until either the function value is less than the error tolerance (indicating that the root is found) or the maximum number of iterations is reached without achieving the required tolerance. The function outputs the final approximation and its corresponding error, along with the number of iterations taken to reach the result.

`newton\_method\_project1` - This function encapsulates Newton's method, a classic root-finding algorithm leveraging calculus to iteratively converge to a solution of f(x) = 0. It requires an initial approximation `x\_0`, and the function `func\_handle` alongside its derivative `func\_prime\_handle`.

At each iteration, the method assesses the current guess by applying the Newton formula: a new guess `x` is computed as the previous `x` minus the quotient of the function value at `x` and the derivative at `x`.

This process is repeated until the change in successive estimates is less than the specified tolerance `e\_a`, or until `i` iterations are completed. The vectors `x\_vals` and `ea\_vals` capture the succession of approximations and their corresponding errors, providing insight into the convergence behavior of the method.

`secant\_method\_project1` - This function implements the secant method, an iterative root-finding technique that approximates solutions to the equation f(x) = 0. Unlike Newton's method, the secant method does not require the calculation of derivatives. It starts with two initial approximations, `x0` and `x1`, and iteratively produces new estimates using the values of the function `func\_str` at these points.

In each iteration, the method uses the previous two estimates to compute a new approximation of the root. This is achieved by calculating the secant line through the points (x0, f(x0)) and (x1, f(x1)) and then finding the x-intercept of this line.

The process continues until the absolute value of the function at the latest estimate is less than the specified tolerance `eps`, or until a maximum number of iterations `max\_iter` has been reached. The vectors `x\_vals` and `ea\_vals` are updated at each iteration with the latest root estimate and its corresponding relative error, respectively.

If the solution is found within the specified tolerance before the maximum number of iterations, the function prints the solution and the number of function calls made. If the process exceeds `max\_iter` without finding an adequate solution, it aborts execution and indicates that a solution was not found.

`modified\_secant\_method\_project1` - This function is a scripted implementation of the Modified Secant method, a numerical technique used for finding roots of a function f(x) = 0. The method is a variation of the Secant method, introducing a small perturbation `delta` to the current estimate `x0` to avoid the need for two initial approximations.

The primary procedure `modified\_secant` takes a function handle `func\_handle`, an initial approximation `x0`, a perturbation factor `delta`, a tolerance `eps`, and a maximum number of iterations `max\_iter`. It starts the root-finding process, iterating to refine the estimate of the root until the absolute value of the function at the current estimate is less than the tolerance, or the maximum number of iterations is reached.

The iterative process consists of generating a perturbed point `x1` from the current estimate `x0`, evaluating the function at both `x0` and `x1`, and then using these values to calculate a new estimate. This new estimate is obtained by taking the current estimate `x0` and adjusting it by a factor related to the function values and the perturbation.

Throughout the iterations, the function maintains arrays `x\_vals` and `ea\_vals` to record the sequence of estimates and their corresponding relative errors, respectively. After each iteration, it updates these arrays with the new estimate and its relative error.

Upon completion of the iteration process, if a solution is found within the specified tolerance, the function prints the number of function calls and the found solution. If the function fails to find a solution within the tolerance after the maximum number of iterations, it prints a message indicating that execution is aborted.

If the function terminates due to an error (such as a zero denominator), it immediately stops execution and prints an error message. If the final function value is above the specified tolerance, the `no\_iterations` is set to -1 to indicate that the desired accuracy was not achieved.

`fzero\_method\_project1` - This function provides a MATLAB interface to the built-in `fzero` function, which is designed to find the roots of a nonlinear equation defined by `f\_func`. The user supplies an initial guess `x0`, a tolerance `tol`, and a maximum number of iterations `max\_iter`.

The function sets up an options structure `options` that specifies the tolerance for the root and the maximum number of iterations the `fzero` function will execute. An output function `@outfun` is also defined and passed to `fzero` via the options structure. This output function is called at each iteration of the root-finding process.

Inside the `outfun`, the relative approximate error `ea` is computed, provided the current estimate `x` is not zero, to avoid division by zero. The value of `ea` is then stored in the array `ea\_vals`, and the current estimate `x` is stored in `x\_vals` for output. The `x\_old` variable is updated with the current estimate `x` for use in the next iteration's error calculation.

After running the `fzero` function with the provided options, the root is stored in the variable `root`, and the total number of iterations taken to find the root is recorded in `iter`.

The function concludes by printing out the found root and the number of iterations required to reach it to the MATLAB Command Window. These print statements are currently uncommented and will execute when the function is called.

By capturing the sequence of estimates and their corresponding errors in `x\_vals` and `ea\_vals`, this function not only finds the root but also provides insight into the convergence behavior of the `fzero` method.

`muller\_project1` - This function implements the Muller method for finding roots of a given function. It is a root-finding algorithm that uses quadratic interpolation to approximate the roots of a real-valued function. The function accepts an initial set of points `p0`, `p1`, `p2`, the function for which the root is sought `func`, a tolerance `TOL`, and a maximum number of iterations `N0`.

The method begins by calculating the differences `h1` and `h2` between consecutive initial points, as well as the divided differences `DELTA1` and `DELTA2`. These are used to compute the coefficient `d` of the quadratic interpolating polynomial.

The algorithm proceeds iteratively, updating the value of `p` (the current estimate of the root) using Muller's method, which involves solving a quadratic equation derived from the interpolating polynomial. The discriminant `D` is calculated, ensuring that the correct sign is chosen to avoid cancellation errors when computing the new estimate `h`.

The estimated root `p` and the absolute error `abs(h)` are then stored in the arrays `estimated\_roots` and `approx\_errors` respectively. If the absolute error is less than the specified tolerance `TOL`, the function displays the estimated root and terminates.

If the tolerance condition is not met, the method updates the points `p0`, `p1`, and `p2` for the next iteration and recalculates the necessary differences and divided differences. This process repeats until the tolerance is met or the maximum number of iterations `N0` is reached.

Should the method exceed `N0` iterations without finding a root within the desired tolerance, it prints a message indicating failure to converge within the specified number of iterations.

The function uses the `format long;` statement at the beginning to ensure that results are displayed with maximum precision, which is particularly useful for observing the behavior of iterative root-finding methods.

# Other Helper Functions

These functions were created for the purpose of aiding the user inserting necessary values for each method in the menu without having complications:

`g\_of\_problem1\_equation` - This function calculates and returns an angle `theta\_0` such that a projectile launched with that angle from an initial height `y\_0` with an initial velocity `v\_0` will reach a target at a horizontal distance `x` and a final height `y`. The calculation is based on the projectile motion equations under uniform gravity `g`:

* It defines the gravitational acceleration `g`, the initial launch velocity `v\_0`, the horizontal distance to the target `x`, the initial height `y\_0`, and the target height `y`.
* Using these parameters, the function calculates the angle `theta\_0` using the inverse tangent function `atan`. This is derived from the kinematic equations of projectile motion, taking into account the initial and final vertical positions of the projectile, the distance traveled, the effect of gravity, and the initial velocity.
* The term `(g \* x^2) / (2 \* v\_0^2 \* (cos(theta\_0))^2)` represents the vertical displacement due to gravity when the projectile has traveled a horizontal distance `x`.
* The angle `theta\_0` is recalculated by rearranging the projectile motion equation to solve for `theta\_0`.

`g\_of\_problem2\_equation` - This function is used to find a new value `x\_new` which is used in a context of fixed-point iteration for an electrical force problem. The function uses constants such as the permittivity of free space `e\_0`, the charges `q` and `Q`, the radius of a ring `a`, and a target force `F\_target`:

* The permittivity of free space `e\_0` is a fundamental physical constant used to calculate electric forces in vacuum.
* The charges `q` and `Q` are the magnitudes of two point charges or distributed charges involved in the problem.
* The constant `a` represents the radius of a ring, suggesting a charge distribution along a circular path.
* `F\_target` is the target electric force that we are trying to achieve.

The equation for `x\_new` is derived from Coulomb's law, adjusted for the geometry of the problem which seems to involve a point charge `q` at a distance `x` from the center of a charged ring with a total charge `Q` and radius `a`. The force `F\_target` is set to be the desired force at this point.

The expression `(x^2 + a^2)^(3/2)` is part of the formula that calculates the electric field due to a ring of charge at a point along the axis passing through the center of the ring, and `x\_new` is calculated such that the electric force `F\_target` would be produced at that point.

In the context of fixed-point iteration, this function is likely called repeatedly with an initial guess for `x`, and the output `x\_new` is then used as the new input for the next iteration, until the value converges to a stable point where `x` does not change significantly between iterations.

`g\_of\_problem3\_equation` - This function generates an anonymous function `fhandle` which calculates the Colebrook equation for fluid flow in a pipe. This equation is commonly used to estimate the friction factor `f` for turbulent flow in a circular pipe based on the Reynolds number `Re`:

* It takes the Reynolds number `Re` as an input, which is a dimensionless quantity in fluid mechanics used to predict flow patterns in different fluid flow situations.
* The function then returns an anonymous function `fhandle`, which calculates the inverse of the square of the friction factor `f`. This is based on an implicit equation from the Colebrook-White formula, which is typically solved using numerical methods such as the Newton-Raphson method or Fixed-Point Iteration.

`problem1\_d\_equation` - This function calculates the derivative of the projectile's motion equation with respect to the launch angle `theta\_0`. It is used to find the angle that would make a projectile hit a certain target point, given initial conditions:

* It defines `g` as the gravitational acceleration, `v\_0` as the initial velocity of the projectile, and `x` as the horizontal distance to the target.
* The function `dy` computes the derivative of the projectile's trajectory equation with respect to `theta\_0`, which is essential for methods like Newton-Raphson that need the derivative to find the root of an equation.

`problem1\_equation` - This MATLAB function defines the equation of the trajectory of a projectile based on the angle of launch (`theta\_0`). It determines the vertical position (`y`) of a projectile at a certain horizontal distance (`x`), considering initial velocity (`v\_0`), gravitational acceleration (`g`), and initial height (`y\_0`):

1. The term tan(theta\_0) \* x calculates the linear increase in height based on the angle theta\_0.
2. The term -(g \* x^2) / (2 \* v\_0^2 \* (cos(theta\_0))^2) represents the parabolic arc due to gravity's influence on the projectile.
3. y\_0 - 1 adjusts the equation based on the initial height from which the projectile is launched, minus 1 meter from the target height to account for some specific condition or requirement in the problem.

`problem2\_d\_equation` - Derivative of the electrostatic force between a point charge and a charged ring.

1. Constants `e\_0`, `q`, `Q`, and `a` are defined for the calculation.
2. `numerator` computes the difference in forces adjusted for distance `x`.
3. `denominator` normalizes the force over the cube of the distance plus the square of the ring radius.
4. `F\_diff\_prime` gives the rate of change of force with respect to `x`.

`problem2\_equation` - This function calculates the difference between the electrostatic force at a point `x` from the center of a charged ring and a target force value.

1. Utilizes the constant for vacuum permittivity `8.9e-12` C²/(Nm²).
2. Charge values `2e-5` C for both the point charge and the ring are used.
3. The radius `a` of the ring is 0.85 meters.
4. `calc` computes the electrostatic force at distance `x` from the ring's center and subtracts the target force of `1.25` Newtons.

`problem2\_possible\_root ` - This MATLAB script is designed to identify intervals where the root of the `problem2\_equation` may exist by checking for sign changes in the function values over a specified range of `x`.

1. `problem2\_equation` is an anonymous function that calculates the difference between electrostatic force at a point and a target force value.
2. `x\_vals` is an array of 1000 linearly spaced points between 0 and 2.
3. `F\_diff\_vals` stores the computed values of `problem2\_equation` for each `x` in `x\_vals`.
4. The script iterates over `F\_diff\_vals` to check for sign changes, indicating potential brackets around roots.
5. When a sign change is detected, it reports the interval as a possible bracket where a root can be found.

The `problem3\_d\_equation` function takes in the Reynolds number (`Re`) and returns an anonymous function `dfhandle`, which represents the derivative of the von Karman equation with respect to the friction factor `f`:

1. The `derivative\_of\_problem3\_equation` is a nested function where the actual derivative calculation is supposed to be implemented based on the von Karman equation for fluid flow in pipes.
2. The placeholder derivative shown in the function `derivative\_of\_problem3\_equation` is just an illustrative example and is not the correct derivative of the von Karman equation.
3. Users must replace the placeholder derivative with the correct derivative expression derived from the von Karman equation.

`problem3\_equation` function is designed to return an anonymous function `fhandle` that captures the Reynolds number `Re` for subsequent evaluations:

1. The inner function `problem3\_equation\_internal` calculates the von Karman equation for pipe flow, which is used to find the friction factor `f` given a Reynolds number `Re`.
2. The left-hand side (`lhs`) of the von Karman equation is `1 / sqrt(f)`, and the right-hand side (`rhs`) is `4 \* log10(Re \* sqrt(f)) - 0.4`.
3. The difference between the left-hand side and right-hand side (`residual`) represents the discrepancy from the root, which the numerical methods aim to minimize or eliminate to find an accurate value of `f`.

`problem3\_possible\_root` - The script visualizes the von Karman equation for a range of Reynolds numbers (`Re\_values`) by plotting the equation values against the friction factor `f`:

1. `Re\_values` are created with a `linspace` function, generating 10 values evenly spaced between 10,000 and 500,000.
2. The plot is initiated with `hold on` to allow multiple plots on the same figure.
3. A for-loop iterates over the `Re\_values`, creating an anonymous function `colebrook\_eq` for each `Re`, which represents the von Karman equation.
4. `f\_vals` is a linear space of 1000 points between 0.001 and 0.01, representing possible friction factor values.
5. `arrayfun` is used to evaluate `colebrook\_eq` over the range of `f\_vals`, and the results are plotted with a label for each `Re`.
6. Inside the loop, another loop checks for sign changes in consecutive `colebrook\_vals`, which indicate potential brackets where the root of the equation might be found, and prints the bracket range.
7. The plot is finalized with a grid, labeled axes, a title, and a legend showing different `Re` values.
8. `hold off` is used to finalize the plot, indicating that no more elements will be added.

# Methodology

## Problem 2

Aerospace engineers sometimes compute the trajectories of projectiles such as rockets. A related problem deals with the trajectory of a thrown ball. The trajectory of a ball thrown by a right fielder is defined by the (x, y) coordinates as displayed in Fig. 1. The trajectory can be model

A number and a line

Description automatically generated with medium confidence

A diagram of a curve

Description automatically generated

1. Find the appropriate initial angle θ0, if v0 = 30 m/s, and the distance to the catcher is 90 m. Note that the throw leaves the right fielder’s hand at an elevation of 1.8 m and the catcher receives it at 1 m. Use 𝜀𝑠 = 0.01%

Given:

v0 = 30/ms

y0 = 1.8m

y = 1m

g = 9.81 m/s2

We can use the following code to find our first initial angle for θ0:

A computer screen shot of a code

Description automatically generated

A computer code with black text

Description automatically generated

Using these as our θ0:

[xL, xu] = [0.661967, 0.663540]

For the Bisection method, we get the following:

A screenshot of a computer

Description automatically generated

We can see here that both the root and the error are able to converge.

For the False Position method, we get the following:

A screenshot of a computer

Description automatically generated

For the fixed-point iteration method, we have to derive our g(x) first before using the method. The following code is used for the g(x) of the equation:

A screenshot of a computer

Description automatically generated

This equation will then be used for the fixed-point iteration method, along side the same initial value for x0:

A screenshot of a computer

Description automatically generated

Moving on, for the Newton method, we first need to find the derivative of the equation. After that, we can plug it into code like so:

A white background with black text

Description automatically generated

Then, we just pass the same initial guess as always:

A screenshot of a computer

Description automatically generated

Using the secant method:

A screenshot of a computer

Description automatically generated

Using Modified Secant, it will work almost like the Secant method, but with a slight perturbation delta:

A screenshot of a computer

Description automatically generated

Using `fzero` from Octave, we have to first modify the options which is rather trivial and then we can use it to get the following plot:

A screenshot of a computer

Description automatically generated

Lastly, the Muller method:

A screenshot of a computer

Description automatically generated

1. Repeat part a using different initial guesses (3 different values where applicable).

Using a different initial guess:

A screenshot of a computer

Description automatically generated

Using a second value closer to the first initial guess using in (a):

A screenshot of a computer

Description automatically generated

1. Plot a graph of the approximation percentage error for all the used algorithms in part a.

All plots in (a) and (b) include Root Estimation plot and Error Approximation plot.

1. Which algorithm is the fastest?

The Newton-Raphson method, generally considered fast for finding roots due to its quadratic convergence, where the number of accurate digits doubles with each iteration. This method uses the function's value and its derivative, providing an efficient prediction of the root, which often results in achieving the desired accuracy with fewer iterations compared to other methods like bisection or secant. The adaptive step size, which adjusts based on the slope of the function, further optimizes the convergence process. However, the method's performance is contingent upon a good initial guess and a well-behaved function; otherwise, it may not converge or could be outperformed by other algorithms.

# Problem 3

A total charge *Q* is uniformly distributed around a ring-shaped conductor with radius *a*. A charge *q* is located at a distance *x* from the center of the ring (Fig. 2). The force exerted on the charge by the ring is given by:

Where . Find the distance where the force is 1.25 N if and are C for a ring with a radius of 0.85 m.

You need to run all codes when possible.

A circle with a line and arrows

Description automatically generated with medium confidence

We begin similarly with Problem 2 by finding possible roots with the following code:

A white background with colorful text

Description automatically generated

Which yields the following possible roots:

A white background with black text

Description automatically generated

[x1, x2] = [0.240240, 0.242242]

[x1, x2] = [1.289289, 1.291921]

Using all numerical methods at once, passing various inputs needed for each piece of code, but specific initial guesses as the ones found, we can see that all solutions converge onto 0.2410~:

A screenshot of a computer

Description automatically generated

# Problem 4

For fluid flow in pipes, friction is described by a dimensionless number, the Fanning friction factor . The Fanning friction factor is dependent on a number of parameters related to the size of the pipe and the fluid, which can all be represented by another dimensionless quantity, the Reynolds number . A formula that predicts given is the von Karman equation:

Typical values for the Reynolds number for turbulent flow are 10,000 to 500,000 and for the Fanning friction factor are 0.001 to 0.01. Develop a function to solve for given a user-supplied value of between 2,500 and 1,000,000. Design the function so that it ensures that the absolute error in the result is E < 0.000005.

For this problem, we had to modify our usual steps to find possible brackets:

A screenshot of a computer code

Description automatically generated

This code, would yield several possible brackets perfect for our methods:

A screenshot of a computer

Description automatically generated

We can use the first bracket and the values given inside the problem description. Namely:

Re = 2,500 through 1,000,000

[x\_1, x\_2] = [0.001, 0.01]

With these values, we can go over to plug them into our numerical methods functions:

A screenshot of a computer

Description automatically generated